The University of South Carolina Mini-Conference on Matrix Theory and Linear Algebra

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1. ACTIVITIES AND PARTICIPANTS

This conference was held during the weekend of 14 and 15 February 1981. The following people participated in the proceedings: George P. Barker (Missouri), Stephen Barnett (Bradford and North Carolina State), Jean H. Bevis (Georgia State), Joel Brawley (Clemson), Ludwig Elsner (Bielefeld and Calgary), Frank J. Hall (Georgia State), Robert Hartwig (North Carolina State), Charles R. Johnson (Maryland and South Carolina), Don Jordan (South Carolina), Thomas L. Markham (South Carolina), Valerie Miller (South Carolina), Michael Neumann (South Carolina), Robert J. Plemmons (Tennessee), George D. Poole (Emporia State), Nicholas J. Rose (North Carolina State), Jeffrey A. Ross (South Carolina), Tim Szeliga (South Carolina), and Don Warner (Clemson). The meeting was organized by Charles R. Johnson, Thomas L. Markham, Michael Neumann, and Jeffrey A. Ross with the enthusiastic support of the Department's Chairman, William T. Trotter.

The conference was conducted in a classroom and there were no formal talks. Instead, the format was to allow each participant who so wished 15–30 minutes to outline problems of present or continuing interest. Each such presentation was followed by a discussion involving the participants. The problems that were raised covered a wide range of topics in matrix theory and reflected well the different areas of interest of the people present. Towards the conclusion of the conference a session was held in which other issues, such as recent or forthcoming publications in linear algebra and related areas,

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forthcoming meetings, etc., were brought by some participants to the attention of the others. The climax of this session was a reading of a poem, "A Matrix Confederacy," written during the two days of the conference by the man from the Brontë country, Stephen Barnett.

South Carolina's warm winter weather undoubtly contributed towards the relaxed atmosphere of the gathering. We wish to thank the College of Science and Mathematics and its dean, Dr. James Durig, for providing funds for two excellent conference meals. In addition, some participants were lodged in Capston House, several floors above a sorority. There was not a complaint registered from any of these people.

It was suggested to report this conference in the Letters and to request the participants to contribute to it research problems which, however, did not have to coincide with the problems raised during the meeting. These research questions are presented below. The conference has already brought tangible benefits. We are glad to report that L. Elsner, C. Johnson, J. Ross, and J. Schönheim have solved a conjuncture and written a paper entitled "On a generalized matching problem arising in estimating the eigenvalue variation of two matrices."

2. RESEARCH PROBLEMS

The references given in each problem relate only to the question under consideration and therefore appear immediately following its text.

On Ratios of the Spectral Norms of Certain Pairs of Matrices

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For two $n \times n$ matrices A and B with eigenvalues λ_i and μ_i, $1 \leqslant i \leqslant n$, respectively, define

$$S_A(B) = \max_{1 \leqslant i \leqslant n} \min_{1 \leqslant j \leqslant n} |\mu_i - \lambda_j|.$$

Upper bounds for $S_A(B)$ have been given, among others, by Bhatia and Friedland [1] and Henrici [2]. The bound due to Bhatia and Friedland is

$$S_A(B) \leqslant (2M)^{1-1/n} n^{1/n} \|A-B\|_2, \tag{1}$$

where $M = \max\{\|A\|_2, \|B\|_2\}$, and the bound due to Henrici is

$$S_A(B) \leqslant \rho \begin{pmatrix} 0 & \Delta & & & & \\ & \cdot & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & 0 & \Delta \\ d & \cdot & \cdot & \cdot & \cdot & d \end{pmatrix}, \tag{2}$$

where $d = \|A-B\|_2$, $\Delta = \Delta_2(A)$ denotes the departure from normality of A (see [2] for precise meaning), and ρ denotes the spectral radius of a matrix. The Bhatia-Friedland bound can be derived from Henrici's result by noting first from $U^*AU = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) + T$, where T is strictly upper triangular, and from $\|T\|_2 = \Delta$ and $\|A\|_2 \leqslant M$. These lead to the inequality

$$\Delta \leqslant 2M,$$

which is then used in the proof that (2) implies (1).

The upper bound (1) could be improved if for each n the quantity

$$\varepsilon(n) = \sup\left\{ \frac{\|T\|_2}{\|D+T\|_2} : T \text{ is strictly upper triangular, } D \text{ is diagonal and } T+D \neq 0 \right\} \tag{3}$$

were to be known. It is possible to show that $1 \leqslant \varepsilon(n) \leqslant 2$, $\varepsilon(2) = 1$ and $\varepsilon(3) = (\frac{4}{3})^{1/2}$. The question of the precise determination of (3) appears to be connected to the behavior of unitary Hessenberg matrices through the following observation. Given an $n \times n$ strictly upper triangular matrix, define the quantity

$$\delta(n) = \sup\left\{ \frac{\|T'\|_2}{\|T' + DJ\|_2} \right\}, \tag{4}$$

where

$$J=\begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 1 & \cdot & & & & \cdot \\ & \cdot & \cdot & & & \cdot \\ & & \cdot & \cdot & & \cdot \\ & & & \cdot & \cdot & \cdot \\ 0 & & & & 1 & 0 \end{pmatrix}.$$

and the supremum is taken over all upper triangular matrices T' and diagonal matrices D for which $T'+DJ\neq 0$. It can be shown that for $n\geqslant 3$,

$$\varepsilon(n)=\delta(n-1).$$

Now if $T'+DJ$ is a unitary (Hessenberg matrix), then in most cases $\|T'\|_2 > \|T'+DJ\|_2$, and for $n=2$ the supremum in (4) is attained by such a pair. I suspect that this is true also for $n>2$.

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Diagonal Symmetrizability and Unary Operations on M-Matrices

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Let M and N denote n-by-n nonsingular *irreducible* M-matrices [3] throughout, and let $\circ$ indicate the *Hadamard* (or componentwise) [1] *product* of two matrices of the same size. We say that an n-by-n real matrix A is *diagonally symmetrizable* if there exists a diagonal matrix D with positive diagonal entries such that AD is symmetric. (It is easy to see that this is equivalent to symmetrizability by diagonal similarity, or by left diagonal multiplication by a positive diagonal matrix.) Computationally, diagonal symmetrizability is not an especially difficult problem, requiring only careful analysis of the relatively simple, and highly redundant, system $AD - DA^T = 0$, linear in the diagonal entries of D, or analysis of the cycle structure of A, for example. We emphasize here, however, a problem associated with the connection between diagonal symmetrizability and the minimum eigenvalue of a certain unary operation applied to M-matrices.

It is known [2] that the matrix $M \circ N^{-1}$ is again an M-matrix, and furthermore that if M is diagonally symmetrizable, then $\lambda_m(M \circ M^{-1}) = 1$, where λ_m denotes the necessarily real and nonnegative "minimum" of the Perron-Frobenius eigenvalue of an M-matrix. The problem we emphasize here is to prove the converse, namely: if $\lambda_m(M \circ M^{-1}) = 1$ then M is diagonally symmetrizable. We also conjecture that, in general, $0 < \lambda_m(M \circ M) \leqslant 1$, with equality being attained in the right-hand inequality if and only if M is diagonally symmetrizable. This would suggest a measure of "near" diagonal symmetrizability, namely $\lambda_m(M \circ M^{-1})$ being large (close to 1). A further question is whether there is a positive minimum (as a function of n) for $\lambda_m(M \circ M^{-1})$ over irreducible M-matrices, and, if so, what its nature is.

A simple calculation yields that for a general n-by-n nonsingular matrix A, the row sums of $A^{-1} \circ A$ are just the diagonal entries of $A^{-1}A^T$ (which is I if and only if A is symmetric). This suggests (upon further calculation) that if $\lambda_m(M \circ M^{-1}) = 1$ and $M \circ M^{-1}x = x$, then the diagonal matrix would symmetrize M would be one whose diagonal entries are the components of x (which may be taken to be positive). An interesting side issue suggested here would be the study of *quasisymmetric matrices*: those nonsingular A for which $A^{-1} \circ A$ has all row sums equal to 1. If A_{ij} is the $(n-1)$-by-$(n-1)$ submatrix of A obtained via deletion of row i and column j, the condition just mentioned is algebraically equivalent to $\Sigma_{j=1}^{n}(-1)^{i+j}\, a_{ij} \det A_{ji} = \Sigma_{j=1}^{n}(-1)^{i+j} a_{ij} \det A_{ij}$, $i = 1, \dots, n$ (the right and hand side is, of course, $\det A$ for each i), except that this condition on A may be studied even in the singular case.

Finally, we note that the study of diagonally symmetrizable M-matrices is entirely general, since the question for an arbitrary real matrix A may be reduced to one for a corresponding M-matrix. First, check to see if $A = (a_{ij})$ is *sign-symmetric* (i.e. a_{ij} and a_{ji} have the same sign, either both positive, both negative, or both zero), and if so, add a sufficiently large multiple of the identity to the *comparison matrix* of A (take absolute values componentwise and insert minus signs off the diagonal) to produce and M-matrix M. Then A will be diagonally symmetrizable if and only if M is.

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Spectrally Dominant Generalized Matrix Norms and Powers Inequalities

Charles R. Johnson

A *generalized matrix norm* (gmn) G is simply a vector norm on $M_n(C)$, the n-by-n complex matrices [1]. Specifically, this means that $G: M_n(C)\to R$ satisfies for all $A, B\in M_n(C)$

$$G(A)\geqslant 0, \quad \text{and} \quad G(A)=0 \text{ iff } A=0, \tag{1}$$

$$G(\alpha A)=\alpha G(A) \qquad \text{for all} \quad \alpha\in C, \tag{2}$$

and

$$G(A+B)\leqslant G(A)+G(B). \tag{3}$$

If, in addition, G satisfies

$$G(AB)\leqslant G(A)G(B) \qquad \text{(multiplicativity)}, \tag{4}$$

then G is called a *matrix norm*.

Let $\rho(A)$, $A\in M_n(C)$, denote the spectral radius (maximum absolute value of the eigenvalues) of A. A gmn G is called *spectrally dominant* if

$$\rho(A)\leqslant G(A) \qquad \text{for all} \quad A\in M_n(C).$$

It is well known that matrix norms are spectrally dominant, but many gmn's which are not matrix norms are also spectrally dominant. Several characterizations of spectrally dominant gmn's have been given [1,2,3], and all (necessarily) involve some form of weakened multiplicativity. In particular, G is a spectrally dominant gmn if and only if for each $A\in M_n(C)$ there is a constant α_A (depending only upon G and A) such that for all positive integers K, $G(A^k)\leqslant \alpha_A G(A)^k$ [3]. The question we emphasize here is whether or not the constant in the theorem just mentioned can be taken to be uniform in (independent of) A. That is, we wish to give an example of spectrally dominant gmn G for which the (minimum) α_A's are unbounded above, or prove that if G is a spectrally dominant gmn, then there is constant α, depending only upon G, such that $G(A^k) \leqslant \alpha G(A)^k$ for all positive integers k and all $A \in M_n(C)$. The problem, of course, is that it is difficult to give any notion of the α_A which is demonstrably continuous in A.

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A Unification Problem Concerning an Inequality of Oppenheim

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1. A Brief Look at the Background of the Problem. Suppose each of A and B is an $n \times n$ matrix with entries from the field of complex numbers. The Hadamard product of A and B is the $n \times n$ matrix $A * B = (a_{ij}b_{ij})$. We shall not make an attempt to define all the terms we use, since most of these terms are standard and are readily accessible in the list of references which we give.

In 1930, Sir Alexander Oppenheim proved the following result.

THEOREM 1 [5, Theorem 2]. *Suppose each of A and B is an $n \times n$ positive semidefinite matrix. Then* $\det(A * B) \geqslant (\prod_{i=1}^n a_{ii}) \det B$.

The author proved the following analogue of Oppenheim's inequality.

THEOREM 2 [4, Theorem 2]. *If each of A and B is an $n \times n$ tridiagonal oscillatory matrix, then* $\det(A * B) \geqslant (\prod_{i=1}^n a_{ii}) \det B$.

Subsequently, the author was able to unify Theorems 1 and 2 via a class of matrices defined by Karl Goldberg [3].

In 1964, Lynn stated another result of this type. For notation, we shall follow that given in [2]. If A is an H-matrix, the $H(A)$ denotes the comparison matrix of A.

THEOREM 3 [2]. *If A and B are $n \times n$ H-matrices, then* $\det H(A * B) \geqslant \prod_{i=1}^{n} |a_{ii}| \det H(B)$.

Finally, Fiedler and Ptak have given yet another analogue.

THEOREM 4 [1, Theorem 4.1]. *Suppose B is an $n \times n$ matrix where $n \geqslant 2$. The following properties of $B = (b_{ij})$ are equivalent:*

(i) *There exists a diagonal matrix D with positive diagonal elements such that the matrix $C = D^{-1} BD$ satisfies $0 \neq |c_{ii}| \geqslant |c_{ik}|$ for all indices i and k.*

(ii) *If A is an H-matrix, then $A * B$ is an H-matrix and*

$$\det(H(A * B)) \geqslant \prod_{i=1}^{n} |b_{ii}| \det H(A).$$

2. *The Problem.* What is the appropriate setting for these results? Is there a unification theorem from which Theorems 1–4 will follow?

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Monotonicity-Type Conditions for Solving Singular Systems by Iterative Methods

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1. *Motivation.* A recent Bell Laboratories report [3], in which congestion and overload in traffic systems are analysed through network control theory, concludes that the use of iterative methods for solving singular consistent systems of linear equations is probably the most feasible technique for obtaining steady-state vectors for large network problems. In earlier works

[3] and [4] (see also [1, Chapters 6–9] where further background material can be found), in which the object of investigation is iterative methods for singular systems, it was suggested that a particularly attractive application of these methods from the computation point of view would, indeed, be the computation of stationary distribution vectors for Markov chains. It is thus interesting to note that the Bell Laboratories report does not mention these earlier publications.

2. *The Problematic.* The works [1], [3], and [4] mentioned above, as well as others (see [2] and [8]), demonstrate that when one tries to extend monotonicity-type results (an $n \times n$ matrix A is said to be *monotone* if for any vector x such that $Ax \geqslant 0$ it follows that $x \geqslant 0$ or, equivalently, A is nonsingular and $A^{-1} \geqslant 0$) for the convergence of iterative schemes for nonsingular systems to the singular case, one of the following situations is encountered:

(1) There is more than a single possibility for generalization. Moreover, while one generalization leads to sufficient conditions for convergence, another will only lead to some necessary ones, but seldom are both present. As an example we cite the different extensions to the notion of *regular splittings* (introduced in [8] for nonsingular systems) considered in [4] and [5].

(2) Loss of necessary conditions for convergence. As an illustration consider the example of a *weak regular splitting* for an M-matrix with "property c" given in [5]. Further, in that paper it is shown that only much stronger assumptions on the coefficients matrix guarantee the existence of some of the more important necessary conditions for convergence. (For nonsingular systems weak regular splittings were introduced in [6].)

(3) Loss of comparison for the asymptotic convergence rate of two nonegative iteration matrices induced by two different splittings of the same coefficients matrix. We mention the rather complete breakdown of the Stein-Rosenberg criterion exhibited in [2].

3. *Possible Resolution.* The generalization to the singular case of regular splitting suggested by Meyer and Plemmons [4] appears demanding in terms of assumptions. However it is felt that a further consideration of these authors' approach may lead to a resultion to some of the abovementioned manifestations. More specifically, answers could be extracted from the material in [4] with the achievement of a better understanding of the relationship between complementary subspaces associated with splitting of matrices possessing some generalized monotonicity properties.

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Nonnegative Cone-Containment Problems

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The research in matrix iterative analysis and solvability of nonnegative linear systems $Ax = b$ has promoted a great deal of study on the ring N_n of nonnegative square matrices. In fact, the amount of "spin off research" on N_n was recently underscored by the text by Berman and Plemmons [1].

In particular, there are several nonnegative-type problems which have appeared in the literature recently which are indeed related. By sharing these problems and offering a few remarks as to how we think they might be interrelated, perhaps someone can formulate the single problem whose solution leads to the solution of all the problems.

PROBLEM 1. Suppose $V=\{v_1, v_2,\dots,v_n\} \subseteq R^n_+$, where R^n_+ is the nonnegative orthant in the real n-dimensional space R^n. Let $C(V)$ denote the polyhedral cone [1, p. 2] generated by nonnegative linear combinations of elements in V. Determine a minimal set of vectors in V, say $V_e = \{v_{k_1}, v_{k_2}, \dots, v_{k_t}\}$, for which $C(V) = C(V_e)$. The unique value of t is called the cone dimension of $C(V)$ [2].

PROBLEM 2. Consider the nonnegative linear system $Ax = b$, $A \geqslant 0$, $b \geqslant 0$, $x \geqslant 0$. Determine when the system is consistent, and whenever it is consistent, determine a solution. See [1] for information which relates λ-monotone matrices, weakly monotone matrices, and nonnegative rank factorizations to nonnegative solutions of $Ax = b$.

PROBLEM 3. Determine whether $A \in N_n$ has a nonnegative rank factorization, and if so exhibit one. See [2] and [3].

PROBLEM 4. $P \in N_n$ is prime if P is not monomial [1, p. 67], and $P = AB$, $A \in N_n$, $B \in N_n$ imply either A or B is a monomial. Determine the primes (or nonprimes) in N_n. See [1, pp. 75–81].

Now each of the four problems described above may be described in terms of "cone containment." In the first problem, the task of determining a minimal set V_e and the cone dimension t amounts to determining a minimal subset of V whose cone contains $C(V)$. Actually, V_e will be unique if V_e contains no two vectors which are scalar multiples of each other.

In the second problem, $Ax = b$ is consistent if the cone generated by the columns of A contains the cone generated by b.

In the third problem, suppose $\operatorname{rank}(A) = t$ and N denotes the rows of A. Then A has a nonnegative rank factorization iff there exists a set M of t nonnegative vectors such that $C(N) \subseteq C(M)$, where the cone dimension of $C(M)$ is t [2].

In the fourth problem, the geometrical or cone-containment interpretation of being prime is this: Suppose R and C are the cones generated by the rows and columns of $A \in N_n$ respectively. Let P denote the cone R^n_+, and let S, W be polyhedral cones such that $R \subset S \subseteq P$ and $C \subset W \subseteq P$. Then A is prime if either S or W must necessarily equal P.

Tam's [4] work should be read to obtain further relationships among the four problems described above.

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